IX. *On Infinite Series. By* Edward Waring, *M. D. F. R. S.* Lucasian *Professor of Mathematics in the University of* Cambridge.

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1. MERCATOR first published the contiuation of the common method of division to an infinite series of terms proceding according to the dimensions of a variable quantity; NEWTON did the same for the common method of extraction of roots. Dr. BARROW before applied the same principles in some easy examples to find the asymptotes of curves.

2. The methods of division and extraction of roots were long before taught; but the continuation of them *in infinitum* would have been useless, as the areæ of curves, whose ordinates are $ax^{\frac{n}{m}}$ (where x denotes the abscis, and a, n, and m invariable quantities) had not been discovered many years before the time of MERCATOR's Publication, and consequently it would have been of little use to transform an ordinate or fluxion, whose area or fluent is unknown, into another form, of which the area, &c. is equally unknown.

3. Sir ISAAC NEWTON extended the rule for raising a binomial (to any affirmative power) to negative powers, the extraction of roots and fractional indexes, by applying the law of the series for affirmative powers to them, and continuing it *in infinitum*. M. DE MOIVRE extracted the root, &c. of a multinomial by a series of a similar nature; but these methods will

will only apply in the moſt ſimple caſes, when not more than one root is to be extracted. In every complicate caſe (*viz.* the extraction of roots of quantities which involve the roots of compound quantities) of irrational quantities, recourſe muſt be had to the old methods of multiplication, diviſion, and extraction of roots.

4. If a root of a complicate irrational quantity be required by a ſeries proceeding according to the dimenſions of x; firſt reduce all the irrational quantities contained under the root by multiplication, diviſion, and extraction of roots into ſerieſes proceeding according to the dimenſions of x, ſo that the terms of the leaſt dimenſions be conſtituted firſt, if an aſcending ſeries be required, and ſo on; and the contrary, if a deſcending; then add the ſeveral ſums together, and extract the root of the reſulting ſum by a ſeries which proceeds according to the dimenſions of x, and it will be the root required.

Ex. Let the value of the quantity $\sqrt[3]{}(x^{-2}+a+\overline{b+cx}\times \sqrt[m]{}(d+ex+\sqrt{}(f+gx^2))+\sqrt{}(h+kx^2))$ be required by an aſcending ſeries; firſt, extract the innermoſt root $\sqrt{}(f+gx^2)=$ $f^{\frac{1}{2}}+\frac{gx^2}{2f^{\frac{1}{2}}}-$&c. which add to the quantity $d+ex$ contained under the ſame root (m), and place the terms of the ſum according to the dimenſions of x, and it becomes $(d+f^{\frac{1}{2}})+ex+\frac{gx^2}{2f^{\frac{1}{2}}}-$&c. of which extract the (m) root, and there reſults

$$(d+f^{\frac{1}{2}})^{\frac{1}{m}}+\frac{1}{m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times ex+\left.\begin{array}{l}\frac{1}{m}\times\frac{1-m}{2m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-2m}{m}}\times e^2\\ +\frac{1}{m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times\frac{g}{2f^{\frac{1}{2}}}\end{array}\right\}x^2+$$

&c. multiply this ſeries into $b+cx$, and thence is derived

$$\left.\begin{array}{l} b\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1}{m}}+\frac{1}{m}b\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1-m}{m}}\times e \\ \quad + \quad c\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1}{m}} \end{array}\right\} x + \&c. = P;$$ extract the root $\sqrt{b+kx^2}=b^{\frac{1}{2}}+\frac{kx^2}{2b^{\frac{1}{2}}}+\&c.=Q$; add the three quantities P, Q, and $x^{-2}+a$ contained under the same root $\sqrt[3]{}$ together, and the series resulting, whose terms are constituted according to the dimensions of x, will be $x^{-2}+(a+b\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1}{m}}+b^{\frac{1}{2}}=A)+(\frac{1}{m}b\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1-m}{m}}\times e+c\times\overline{d+f}^{\frac{1}{2}\cdot\frac{1}{m}}=B)\,x+\&c.$; of which extract the cube root $\sqrt[3]{}$, and it will be $x^{-\frac{2}{3}}+\frac{1}{3}Ax^{\frac{4}{3}}+\frac{1}{3}Bx^{\frac{7}{3}}+\&c.$ the root required.

5. The principal use of reducing quantities into series proceeding according to the dimensions of the variable quantity is as before mentioned for finding the area of a curve from its ordinate; or, which corresponds, the integral from its nascent or evanescent increment; but the serieses deduced should converge, otherwise from them cannot be found the area or integral. In the Meditationes Analyticæ a method was first published of finding when these series will converge and when not, *e. g.* the series $a+bx+cx^2+dx^3+\&c.=\int(A+Bx+Cx^2+Dx^3+\&c.)^m\times\dot{x}=P$ will converge when (x) either affirmative or negative is less than the least root (α) of the equation $A+Bx+Cx^2+Dx^3+\&c.=0$, if the roots are possible. A similar rule is given when some of the roots are impossible. The series will diverge when x is greater than α, and the cases are given in which it will converge when $x=\alpha$. The series descending according to the dimensions of x will converge when x is greater than the greatest root of the equation, &c. These principles are fur-

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ther applied in the ſame Book to complicate irrational algebraical functions of x, &c.

Hence moſt commonly the ſeries for the area contained between two ordinates, or integral between two different increments deduced by the common method will diverge; on which account, in the ſame Book, is given a method by interpolation of finding the area or integral contained between any two different values of x by converging ſeries, if the area, &c. is finite.

6. To find whether a given value $(+a)$ is leſs than the leaſt affirmative or negative root (x) of a given algebraical equation $A+Bx+Cx^2+Dx^3+\&c.=0$, if all its roots are poſſible; transform the equation into another, whoſe root z is the reciprocal of the root $x=\frac{1}{z}$ of the given equation, and for z in the reſulting equation write reſpectively $v+a$ and $v-a$; and if from the former ſubſtitution all the terms become negative or affirmative, and from the latter they become alternately negative and affirmative, then will a be leſs than the leaſt root of the given equation. If in the ſame manner, in the given equation for x be ſubſtituted $v+a$ and $v-a$, and the terms reſult as before, then will (a) be greater than the greateſt root affirmative or negative of the given equation.

7. When the integral of an algebraical quantity, whoſe increments are finite, is required; firſt, by the method given in Medit. Analyt. inveſtigate the integral in finite terms, if it can be expreſſed by them; but if not, reduce it into infinite ſerieſes of which the integral of each of the terms can be found, and alſo the ſerieſes for finding the integral contained between the two different given values of the variable quantity may converge.

Serieſes of this kind have been given in the Medit. Analyt. and innumerable of a like kind may be added for finding integrals

integrals by converging ſerieſes either aſcending or deſcending, of which the given increments are either finite or evaneſcent.

7. 2. It may be obſerved, that generally the particular caſe of which the increments are naſcent or evaneſcent may be deduced from the general, in which the increments are finite; and conſequently in many caſes the general will, *mutatis mutandis*, correſpond to the particular; *e. g.* 1. the integral cannot be expreſſed in finite algebraical terms, when any factor in the denominator of the increment has not a ſucceſſive correſpondent one; which is analogous to the caſe of the ſimple diviſor in the denominator of a fluxion publiſhed in the Quadrature of Curves. 2. Nor can it be expreſſed by the above-mentioned terms, when the dimenſions of the variable quantity in the denominator exceed its dimenſions in the numerator by unity, which correſponds to a ſimilar caſe in fluxions firſt given in Medit. Analyt. To theſe may ſeveral others be added.

8. Let the fluent of the fluxion $(A + Bx^n + Cx^{2n} + \&c.)^m \times x^{\theta-1}\dot{x} = ax^\theta + bx^{\theta+n} + cx^{\theta+2n} + \&c. = (A + Bx^n + Cx^{2n} + \&c.)^{m+1} \times \overline{\alpha x^\theta + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.} = a'x^\lambda + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c. = (A + Bx^n + Cx^{2n} + \&c.)^{m+1} \times \overline{\alpha' x^{\lambda'} + \beta' x^{\lambda'-n} + \gamma' x^{\lambda'-2n} + \&c.}$

1. If the infinite aſcending ſeries $\alpha x^\theta + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.$ converges, then will the ſeries $ax^\theta + bx^{\theta+n} + cx^{\theta+2n} + \&c.$ converge, and nearly in the ſame ratio; and *vice verſâ*, if the former diverges, the latter will alſo diverge. In the ſame manner if the infinite deſcending ſeries $\alpha' x^{\lambda'} + \beta' x^{\lambda'-n} + \gamma' x^{\lambda'-2n} + \&c.$ converges, then will the ſeries $a'x^\lambda + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c.$ converge; and if the former diverges, then will the latter alſo diverge; and in all the caſes nearly in the ſame ratio, except only when their convergency is the leaſt.

2. If the feries $ax^{\theta}+bx^{\theta+n}+cx^{\theta+2n}+$&c. converges, then will the feries $a'x^{\lambda}+b'x^{\lambda-n}+c'x^{\lambda-2n}+$&c. diverge, unlefs in cafes of the floweft convergency, where $x=\pm v\sqrt[r]{\pm 1}$, and all the roots of the given equation are of the formula $\pm v\sqrt[s]{\pm 1}$; when $x=\pm v\sqrt[r]{\pm 1}$, the fucceffive terms of the infinite feries deduced from algebraical quantities by the preceding method will ultimately, that is, at an infinite diftance come more near to a ratio of equality with each other than any affignable difference.

3. If the fluxion be $(A+Bx^{n}+Cx^{2n}\ldots\ldots x^{rn})^{m}\times x^{\theta-1}\dot{x}$, then will $\lambda=mrn+\theta$, and $\lambda'=\theta-rn$.

Many more propofitions concerning infinite feries and their convergency are given in the Medit. Analyt.

4. Let the fluxion be $\overline{a+bx^{n}}^{m}\times x^{n}\dot{x}$, of which find the fluent in a feries afcending according to the dimenfions of x^{n}, and it will be

$$a^{m}\times x^{b+1}\left(\frac{1}{b+1}+\frac{m}{b+n+1}\times\frac{b}{a}x^{n}+m\,.\,\frac{m-1}{2(b+2n+1)}\times\frac{b^{2}}{a^{2}}x^{2n}+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3(b+3n+1)}\times\frac{b^{3}}{a^{3}}x^{3n}+\text{&c.}\right)=\frac{b^{m}x^{mn+b+1}}{mn+b+1}+\frac{mb^{m-1}ax^{mn+b+1-n}}{mn+b+1-n}+m\,.\,\frac{m-1}{2}\times\frac{b^{m-2}a^{2}x^{mn+b+1-2n}}{mn+b+1-2n}+\text{&c.}=\overline{a+bx^{n}}^{m+1}$$

$$\times\left(\frac{x^{b+1}}{\overline{b+1}a}-\frac{\overline{m+1}(n)+b+1}{\overline{b+1}\,.\,\overline{b+1+n}}\times\frac{b}{a^{2}}x^{b+n+1}+\frac{(\overline{m+1}(n)+b+1).(\overline{m+2}(n)+b+1)}{\overline{b+1}\,.\,\overline{b+1+n}\,.\,\overline{b+1+2n}}\times\frac{b^{2}}{a^{3}}\times x^{b+2n+1}-\frac{(\overline{m+1}\,.\,n+b+1)\,.\,(\overline{m+2n}+b+1)\,.\,(\overline{m+3n}+b+1)}{\overline{b+1}\,.\,\overline{b+1+n}\,.\,\overline{b+1+2n}\,.\,\overline{b+1+3n}}\times\frac{b^{3}}{a^{4}}x^{b+3n+1}+\text{&c.}\right)=(bx^{n}+a)^{m+1}\times\left(\frac{1}{(mn+b+1)b}\times x^{b+1-n}-\frac{\overline{b+1-n}a}{\overline{mn+b+1}\,.\,\overline{mn+b+1-n}b^{2}}\times x^{b+1-2n}+\frac{\overline{b+1-n}\times\overline{b+1-2n}\times a^{2}}{\overline{mn+b+1}\times\overline{mn+b+1-n}\times\overline{mn+b+1-2n}\times b^{3}}\times x^{b+1-2n}-\text{&c.}\right)\ \text{(L).}$$

The first

firſt ſeries, which aſcends according to the dimenſions of x^n, terminates, when $\frac{mn+h+1}{n}$ is a whole negative number; the ſecond, which deſcends according to the dimenſions of the ſame quantity x^n, terminates when $\frac{h+1}{n}$ is a whole affirmative number. The fluent will terminate both ways when $\frac{h+1}{n}$ is a whole affirmative and m a negative number greater than $\frac{h+1}{n}$.

5. When m is a whole number or $=0$, the fluent $\int \overline{a+bx^n}^m x^h \dot{x}$ can always be found in finite terms of x, or in the above-mentioned finite terms, together with the log. of x; which appears from reducing $\overline{a+bx^n}^m$ into ſimple terms $a^m + ma^{m-1} bx^n + m \cdot \frac{m-1}{2} a^{m-2} b^2 x^{2n} + \&c.$, and multiplying them into $x^h \dot{x}$, and finding the fluents of the reſulting fluxions: but the ſeries found by the preceding method will not always terminate when m is a whole number, and the ſeries findable as above mentioned; when properly corrected, it may be rendered findable, or, which means the ſame thing, the ſeries may be made to terminate.

6. When the ſeries which expreſſes the fluent of a fluxion terminates, we may begin either from one end of the ſeries, or the other; for example, in finding the fluent of the fluxion $\overline{a+bx^n}^m \times x^h \dot{x}$, either aſſume the ſeries $\overline{a+bx^n}^{m+1} \times (\alpha x^{h+1} + \beta x^{h+n+1} + \gamma x^{h+2n+1} + \&c.)$ or the ſeries $\overline{a+bx^n}^{m+1} \times (A + Bx^n + Cx^{2n} + \&c.)$; the former, as is before mentioned, terminates when

when $\frac{mn+h+1}{n}$ is a whole negative number; the latter when $\frac{h+1}{n}$ is a whole affirmative number.

In like manner assume for a descending series $(bx^n + a)^{m+1} \times (\alpha' x^{h+1-n} + \beta' x^{h+1-2n} + \gamma' x^{h+1-3n} + \&c.)$, or $(bx^n + a)^{m+1} \times (Ax^{-\overline{m+1}n} + Bx^{-\overline{m+2}n} + Cx^{-\overline{m+3}n} + \&c.)$; the former will terminate when $\frac{h+1}{n}$ is a whole affirmative; the latter when $\frac{mn+h+1}{n}$ is a whole negative number.

It appears, therefore, that a series will terminate equally by an ascending or descending series; and the end of the one ascending series is the beginning of its correspondent descending one. All these serieses, which do not terminate, proceed on *in infinitum*; one term in the former series becomes infinite, when $h+1+zn=0$; and in the latter (L) when $mn+h+1-zn=0$, z being a whole affirmative number.

10. It has been observed in the Medit. Analyt. that if some quantities contained in the given irrational ones are much less or greater than the rest, it may be preferable in the former case to reduce them into serieses not proceeding according to the dimensions of x, but according to the dimensions of those quantities; and in the latter case according to the reciprocal dimensions of them; and particularly so if the fluent or integral of the terms of the resulting serieses can be found in finite terms, or by tables already calculated.

From similar principles to those before given may be found when the resulting series will converge, and when not.

This method will in many problems be useful, when the value of a near approximate is known.

 Of

Of this caſe I ſhall ſubjoin a few examples, of which ſome have been already publiſhed in the Medit. Analyt.

Ex. 1. Let the ſine and coſine of a given arc A be reſpectively s and c; then will the ſine and coſine of an arc $A+e$, where e bears a very ſmall ratio to A, be reſpectively $s+\frac{ce}{r}-\frac{s}{1\cdot 2r^2}e^2-\frac{ce^3}{1\cdot 2\cdot 3r^3}+\frac{s}{1\cdot 2\cdot 3\cdot 4r^4}e^4+\&c.$ (the ſigns proceed by pairs alternately negative and affirmative) $= s\left(1-\frac{1}{1\cdot 2r^2}\times e^2+\frac{1}{1\cdot 2\cdot 3\cdot 4r^4}\times e^4-\&c.\right)+c\left(\frac{e}{r}-\frac{e^3}{1\cdot 2\cdot 3r^3}+\frac{e^5}{1\cdot 2\cdot 3\cdot 4\cdot 5r^5}-\&c.\right)$; and $c-\frac{s}{r}e-\frac{1}{1\cdot 2r^2}ce^2+\frac{1}{1\cdot 2\cdot 3r^3}\times se^3+\&c.$ (the ſigns proceed as before by pairs alternately negative and affirmative) $=c\left(1-\frac{1}{1\cdot 2r^2}e^2+\frac{1}{1\cdot 2\cdot 3\cdot 4r^2}e^4-\&c.\right)-\frac{s}{r}\left(e-\frac{1}{2\cdot 3r^2}e^3+\frac{1}{2\cdot 3\cdot 4\cdot 5r^4}e^5-\&c.\right)$

This ſeries can alſo eaſily be derived from NEWTON's ſeries by plane trigonometry, and will converge much ſwifter than the ſeries $A-\frac{A^3}{1\cdot 2\cdot 3r^2}+\frac{A^5}{2\cdot 3\cdot 4\cdot 5r^4}-\&c.$ &c. if e bears a ſmall proportion to A.

Ex. 2. Let e be a ſmall quantity in proportion to t, and the given fluxion $\frac{\dot{t}}{r^2+(t+e)^2}=\frac{\dot{t}}{r^2+t^2}-\frac{2te+e^2}{(r^2+t^2)^2}\dot{t}+\frac{(2te+e^2)^2}{(r^2+t^2)^3}\times\dot{t}-\frac{(2te+e^2)^3}{(r^2+t^2)^4}\times\dot{t}+\&c.=\frac{\dot{t}}{r^2+t^2}-\frac{2t\dot{t}}{(r^2+t^2)^2}\times e-\frac{(\overline{r^2+t^2}-2^2\times t^2)\dot{t}}{(r^2+t^2)^3}\times e^2+2t\times\frac{(2\overline{r^2+t^2}-2^2t^2)\dot{t}}{(r^2+t^2)^4}\times e^3+\&c.$ (P); the co-efficient of a term $\dot{t}\times e^{2m}$ of this ſeries will be a fraction whoſe numerator is $\overline{r^2+t^2}^m-\overline{m+1}\cdot\frac{m}{2}\times 2^2\times t^2\times\overline{r^2+t^2}^{m-1}+\overline{m+2}\cdot\frac{m+1}{2}\cdot\frac{m}{3}\cdot\frac{m-1}{4}\times 2^4\times t^4\times\overline{r^2+t^2}^{m-2}-\overline{m+3}\cdot\frac{m+2}{2}\cdot\frac{m+1}{3}\cdot\frac{m}{4}\cdot\frac{m-1}{5}\cdot\frac{m-2}{6}$

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$\times 2^6 \times t^6 \times \overline{r^2+t^2}^{n-3}$ + &c., and denominator $(r^2+t^2)^{2m+1}$; and the co-efficient of a term $\dot{t} \times e^{2m+1}$ will be a fraction whose numerator is $2t \times (\overline{m+1} \cdot \overline{r^2+t^2}^m - \overline{m+2} \cdot \frac{m+1}{2} \cdot \frac{m}{3} \times 2^2 \times t^2 \times \overline{r^2+t^2}^{m-1} + \overline{m+3} \cdot \frac{m+2}{2} \cdot \frac{m+1}{3} \cdot \frac{m}{4} \cdot \frac{m-1}{5} \times 2^4 t^4 \times \overline{r^2+t^2}^{m-2} -$ &c.) and denominator $(r^2+t^2)^{2m+2}$; m being a whole number. The continuation of these serieses is too evident to need enunciation, as must generally be the case when the number of factors contained in the successive terms continues the same or increases in arithmetical progression; and the factors themselves increase or diminish in an arithmetical progression: the terms proceed alternately + and − by pairs.

2. 2. Let the given fluxion be $\frac{\dot{e}}{r^2+(t+e)^2} = \frac{\dot{e}}{r^2+t^2} - \frac{\overline{2te+e^2} \times \dot{e}}{(r^2+t^2)^2} + \frac{(2te+e^2)^2 \times \dot{e}}{(r^2+t^2)^3}$ − &c. $= \frac{\dot{e}}{r^2+t^2} - \frac{2t\dot{e}}{(r^2+t^2)^2} \times e - \frac{(r^2+t^2-2^2t^2)\dot{e}}{(r^2+t^2)^3} \times e^2$ + &c. (Q); this series becomes the same as the preceding by substituting in it $\dot{e}$ for $\dot{t}$; in this series e is considered as variable, in the preceding t.

The fluent of the former series (P) is $\int \frac{\dot{t}}{r^2+t^2} + \frac{e}{r^2+t^2} - \frac{t}{(r^2+t^2)^2} \times e^2 - \frac{(r^2+t^2-2^2t^2)}{3(r^2+t^2)^3} \times e^3$ − &c.; the fluent of the latter series (Q) is the same as of the former (P) except the first term $\int \frac{\dot{t}}{r^2+t^2}$: the co-efficient of the term e^h in the fluents will be the same as the co-efficient of the term e^{h-1} in the serieses (P) and (Q) divided by h.

This series, when the tangent of a given arc is known, finds the arc whose tangent differs very little from the given tangent.

11. Let t be increased or diminished by a quantity e in (P), where e bears a very small ratio to any root or value of t in the

the equations $P=0$ or $\frac{1}{P}=0$; and the resulting quantity expanded into a series $a+be+ce^2+de^3+$ &c. proceeding according to the dimensions of e, and this series be multiplied respectively into $\dot{t}$ and $\dot{e}$; and the fluents of the resulting fluxions found; then will the former differ from the latter by $\int a\dot{t}$; for the former will be $\int a\dot{t}+ae+\frac{\dot{a}}{2\dot{t}}\times e^2+\frac{\ddot{a}}{2.3\dot{t}^2}\times e^3+\frac{\dddot{a}e^4}{2.3.4\dot{t}^3}+$ &c., and the latter $ae+\frac{\dot{a}}{2\dot{t}}e^2+\frac{\ddot{a}e^3}{2.3\dot{t}^2}+$ &c.

2. In the same manner, if more than one variable quantities x, y, z, &c. are contained in (P), which are increased by small increments or decrements α, β, γ, &c., may the increments or decrements of the quantity (P) be deduced from the incremential theorem.

Ex. Let the quantity (P) be $(a+bx^n)^m$, and a, b, x, n, and m be increased by very small increments α, β, γ, δ, and ε; then will $(a+\alpha+\overline{b+\beta}\times\overline{x+\gamma}^{n+\delta})^{m+\varepsilon}=(a+bx^n)^m+(a+bx^n)^m\times$ log. $\overline{a+bx^n}\times\varepsilon+m\times(a+bx^n)^{m-1}\times(\alpha+\beta x^n+b(x^n\delta\times\log. x+nx^{n-1}\gamma))+$ &c.

The terms are to be so placed, according to the dimensions of the increments or decrements, that the greatest may first occur.

12. Let some compound quantities be increased by any small quantities variable or invariable, but not the variable quantities contained in them; then reduce the given quantity into a series proceeding according to the dimensions of the small quantities, and find the fluent, integral, &c. as required.

Ex. Let the given quantity be $\frac{x^m\dot{x}}{(x^n+a^n+\beta)^r}=\frac{x^m\dot{x}}{(x^n+a^n)^r}-$

$r\beta$

$\frac{r\beta\dot{x}x^m}{(x^n+a^n)^{r+1}} + r \cdot \frac{r+1}{2} \times \frac{\beta^2\dot{x}x^m}{(x^n+a^n)^{r+2}} - r \cdot \frac{r+1}{2} \cdot \frac{r+2}{3} \times \frac{\beta^3\dot{x}x^m}{(x^n+a^n)^{r+3}} +$ &c., where the quantity $x^n + a^n$ is increaſed by a ſmall quantity β; then will $\int \frac{x^m\dot{x}}{(x^n+a^n+\beta)^r} = \int \frac{x^m\dot{x}}{(x^n+a^n)^r} - \frac{\beta}{n \times a^n}\left(\frac{x^{m+1}}{(x^n+a^n)^r} - \overline{m+1-rn} \times P\right) + \frac{r\beta^2}{2 \cdot n \times a^n}\left(\frac{x^{m+1}}{(x^n+a^n)^{r+1}} - (m+1-\overline{r+1}n) \times Q\right)$ $\frac{r \cdot \overline{r+1}\beta^3}{2 \cdot 3 \cdot n \times a^n}\left(\frac{x^{m+1}}{(x^n+a^n)^{r+2}} - (m+1-\overline{r+2}n) \times R\right) + \frac{r \cdot \overline{r+1} \cdot \overline{r+2} \cdot \beta^4}{2 \cdot 3 \cdot 4 \cdot n \times a^n}$ $\left(\frac{x^{m+1}}{(x^n+a^n)^{r+3}} - (m+1-\overline{r+3}n) \times S\right)$ &c. The continuation of the ſeries is evident; the letters P, Q, R, S, &c. denote the fluents $\int \frac{x^m\dot{x}}{(x^n+a^n)^r}$, $\int \frac{x^m\dot{x}}{(x^n+a^n)^{r+1}}$, $\int \frac{x^m\dot{x}}{(x^n+a^n)^{r+2}}$, &c.

From the length of the arc of an hyperbola or ellipſe given may be deduced by this ſeries the length of a correſpondent arc of an hyperbola or ellipſe, of which the equations expreſſing the relation between its abſciſſæ and ordinates differ only by very ſmall quantities from the equations expreſſing the relation between the abſciſſæ and ordinates of the former.

13. The ſame principles may be applied to the reſolution of algebraical, fluxional, incremential, &c. equations.

Ex. Let LmM, &c. (Tab. IV. fig. 1.) be a given circle, whoſe center is C and radius (r), and it be required to find an arc LM, ſo that the area LSM deſcribed round a given point S contained between the lines LS, SM, and the arc of the circle LM be equal to a given area (α).

Find an arc Lm = A nearly equal to the arc required LM, of which to radius 1 ſubſtitute s for the ſine, and c for the coſine, and write e for mM = LM − Lm and b for SC; then will LM $\times \frac{r}{2}(\overline{A+e} \times \frac{r}{2}) \pm$ SC $\times \frac{r}{2} \times$ ſin. s arc : LM$\left(\frac{b}{2} \times (A+e-\right.$

A + e

$\frac{\overline{A+e}^3}{2\,.\,3r^2}+\frac{\overline{A+e}^5}{2\,.\,3\,.\,4\,.\,5r^4}$ &c.)) $=\frac{A}{2}\times r\pm\frac{b}{2}\times rs+\frac{r}{2}\pm\frac{b}{2}\times c\times e\mp$ $\frac{b}{2r}\times\left(\frac{1}{2}\times se^2+\frac{1}{2\,.\,3r}\times ce^3-\frac{1}{2\,.\,3\,.\,4r^2}\times se^4-\frac{1}{2\,.\,3\,.\,4\,.\,5r^3}\times ce^5+\right.$ &c.$)=\alpha$; in this equation for $\frac{2\alpha-Ar\mp b\times rs}{r\pm bc}$ substitute π, and for $r\pm bc$ write t, and the equation resulting will be $e\mp$ $\frac{b}{2tr}se^2\mp\frac{b}{1\,.\,2\,.\,3tr^2}\times ce^3\pm\frac{b}{1\,.\,2\,.\,3\,.\,4tr^3}\times se^4\pm\frac{b}{1\,.\,2\,.\,3\,.\,4\,.\,5tr^4}\times ce^5$ $\mp$ &c. $=\pi$. From it find e in terms proceeding according to the dimensions of π, and there results $e=\pi\pm$ $\frac{bs}{1\,.\,2tr}\times\pi^2\pm\frac{3b^2s^2\pm bct}{2\,.\,3t^2r^2}\times\pi^3\pm$&c.

This method of resolving KEPLER's and other problems of cutting a given area described round a point, whether focus or not, in a circle, when an approximate sufficiently near to the area to be found is given, will converge as swift as any known method.

The resolution of this problem may be deduced somewhat different by the following methods. Let the letters b, r, α, denote the same quantities as before, and s be the sine of the arc Lm to radius 1, and $s+o$ the sine of the arc LM nearly $=$ Lm; then will $\frac{r^2}{2}\int\frac{\dot{s}}{\sqrt{(1-\overline{s+o}^2)}}\pm\frac{br}{2}\times\overline{s+o}=\alpha$; reduce the fluxion $\frac{\dot{s}}{\sqrt{(1-\overline{s+o}^2)}}$ into a series $P\dot{s}+Qo\dot{s}+Ro^2\dot{s}+So^3\dot{s}+$ &c. and find the fluents of the fluxions $Q\dot{s}$, $R\dot{s}$, $S\dot{s}$, &c., which let be B, C, D, &c., there will result the equation $o+\frac{C}{t}o^2+\frac{D}{t}o^3+$&c. $=$ $\frac{2\alpha\mp brs-r\times A}{t}=\pi$, where $t=Br^2\pm br$ and $A=$arc Lm; find o in terms of a series proceeding to the dimensions of π, and consequently $s+o$ the sine of the arc LM required.

From

From ſimilar principles may be found the tangent of the arc required from the equation $\frac{r^2}{2}\int \frac{\dot{i}}{1+(t+i)^2} \pm \frac{\overline{t+i} \times r \times b}{2\sqrt{(1+\overline{t+i}^2)}} = \alpha$, where the tangent t to rad. 1 of the arc $Lm = A$ is given. Let the fluent of the fluxion $\left(\frac{\dot{i}}{1+(t+i)^2}\right) = A + Bi + Ci^2 + Di^3 + \&c.$; and $\frac{\overline{t+i} \times b}{\sqrt{(1+\overline{t+i}^2)}} = A' + B'i + C'i^2 + D'i^3 + \&c.$, the equation will become $i + \frac{rC \pm C'}{rB \pm B'} i^2 + \frac{rD \pm D'}{rB \pm B'} \times i^3 + \&c. = \frac{2\alpha - r^2 A \mp r A'}{(r^2 B \pm r B')} = \pi$; from this equation inveſtigate $i = \pi + P\pi^2 + Q\pi^3 + \&c.$, and thence $t + i$ the tangent required.

In like manner may be found the ſecant, coſine, &c. of the arc required.

14. The ſame principles may be applied to cut an area deſcribed round any given point in a given curve equal to an area α.

Let x be the abſciſs and y the ordinate of the given curve, and b the diſtance of the beginning of the abſciſs from the given points, and let (A) be the area of the curve deſcribed round the point S, when the abſciſs is x, which differs very little from the given area (α); to find the value $x + e$ of the abſciſs, when the area $= \alpha$.

Let $y = X$ a function of x, and in X for x write $x + e$, and reduce the reſulting quantity (X′) into a ſeries $X + Be + Ce^2 + De^3 + \&c.$ proceeding according to the dimenſions of e; then will the area $\int y\dot{x} = \int X\dot{x} + e\int B\dot{x} + e^2\int C\dot{x} + e^3\int D\dot{x} + \&c. = A + he + ke^2 + le^3 + \&c.$, and conſequently $A + he + ke^2 + le^3 + \&c. \pm (b \pm \overline{x+e}) \times \frac{y}{2} = A + he + ke^2 + le^3 + \&c. + \frac{1}{2}(b \pm \overline{x+e}) \times (X + Be + Ce^2 + De^3 + \&c.) = A + \frac{1}{2}(b \pm x) \times X + (h + \frac{1}{2}\overline{b \pm x} \times$

$B + \frac{1}{2}X) e + (k + \frac{1}{2}(b \pm x)C + \frac{1}{2}B) e^2 + (l + \frac{1}{2}\overline{b \pm x} \times D + \frac{1}{2}C) e^3 +$ &c. $= \alpha$; find the value of e in a ſeries proceeding according to the dimenſions of $\frac{\alpha - A - \frac{1}{2}(b \pm x) \times X}{b + \frac{1}{2}\overline{b \pm x} \times B + \frac{1}{2}X}$, and $x + e$ will be the abſciſs required; X denotes the value of y, when the abſciſs $= x$.

From the ſame principles may the ordinate y, &c. be found.

This problem may be reſolved in the ſame manner, when X denotes an infinite ſeries deduced from an equation expreſſing the relation between the abſciſs and ordinate of the given curve.

If the given area α be the difference between two areas SPM (α') and SPQ $= \beta$ (fig. 2.); for α ſubſtitute $\alpha' - \beta$, and the operation will be the ſame as the preceding.

15. 1. Given any equations, of which the increments of the quantities contained in them can be found from each other, and given approximate values of each of the unknown quantities, which nearly correſpond to each other; to find approximations, which differ leſs from the quantities themſelves than the given ones.

Suppoſe each of the given approximate values to be increaſed or diminiſhed by ſmall increments or decrements, as e, o, i, &c. which are the approximations to be found; and from the given find the equations reſulting from this hypotheſis; and from theſe may be deduced, by ſimple equations, the approximations ſought e, i, o, &c. by neglecting in them all the powers of e, o, i, &c. except the ſimple ones, and all the products of them multiplied into each other; and conſequently the equations deduced will contain only given quantities and ſimple powers of the unknown e, i, o, &c. to be found.

2. When two or more (n) values of one (x) of the unknown quantities are nearly equal to its given approximate; then the

equation which finds the approximate to x will be a quadratic or an equation of n dimenſions.

3. The approximations found by this method will converge more or leſs, according as the approximations given are more or leſs near to one value of each of the unknown quantities than to the remaining ones, &c.

Theſe principles were printed in the Medit. Algebr. in the years 1768 and 1769.

LEMMA.

15. Let kab (fig. 3.) be a circle, whoſe center is o; and Po perpendicular to the plane of the circle; and the force of any corpuſcle in the circle on the particle P vary as the corpuſcle divided by the (n) power of its diſtance from the particle; to find the attraction of the circle kab on the particle P.

From the ſuppoſita the force of any ring contained between the neareſt concentric circles lcd and hef of which the center is o, on the particle P, will be as the area of the ring divided by the nth power of the diſtance Ph; and conſequently if $Po = A$, $ob = v$, $hl = \dot{v}$, and $p =$ periphery of a circle of which the radius is 1; the attraction of the ring on P will be as $\frac{p \times v\dot{v}}{(A^2+v^2)^{\frac{n}{2}}}$, and the force of it in the direction Po as $\frac{Po}{Ph} \times \frac{pv\dot{v}}{(A^2+v^2)^{\frac{n}{2}}} = \frac{pAv\dot{v}}{(A^2+v^2)^{\frac{n+1}{2}}}$; of which fluxion the fluent is $\frac{pA}{\overline{1-n} \times (A^2+v^2)^{\frac{n-1}{2}}}$; and thence the force of the circle kab on the particle P will be as $\frac{p \times A}{\overline{1-n}Pk^{n-1}} - \frac{p \times A}{\overline{1-n}PO^{n-1}} \left(\frac{p}{\overline{1-n}A^{n-2}}\right)$.

 Cor.

Cor. 1. The force of the area contained between the two circles *kab* and *lcd* on the particle P will be as $\frac{pA}{\overline{1-n}Pk^{n-1}} - \frac{pA}{\overline{1-n}Pl^{n-1}}$.

15. Let a solid be generated by the rotation of a given curve round a line in the axis P*o* or P*o* produced, and A the distance of P from *o*, and x=distance of any circle generated by the rotation of a point in the curve round the axis P*o* from *o*; and y the ordinate to the given curve, of which the abscis is x; and the same things be supposed as before; then will the attraction of the solid on the point P vary as

$$\int \frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}(A-x)^{n-1}}\left(\frac{p\dot{x}}{\overline{n-1}(A-x)^{n-2}}\right) - \int \frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}\times\left(\overline{A-x}^2+y^2\right)^{\frac{n-1}{2}}}.$$

Let the given curve be a circle, of which the center is o, and radius t, and consequently $y^2=t^2-x^2$, and $\int \frac{p\dot{x}}{\overline{n-1}(A-x)^{n-2}}$

$$-\int \frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}\,.\left(\overline{A-x}^2+y^2\right)^{\frac{n-1}{2}}} = \frac{p}{\overline{n-1}\,.\,\overline{n-3}\times(A-x)^{n-3}} -$$

$$\int \frac{p\dot{x}\times\overline{A-x}}{\overline{n-1}\times\left(A^2+t^2-2Ax\right)^{\frac{n-1}{2}}} = \frac{p}{\overline{n-1}\,.\,\overline{n-3}(A-x)^{n-3}} - \frac{p}{n-1}\left(\frac{1}{n-3}+\right.$$

$$\left.\frac{A^2+t^2}{\overline{n-5}\,.\,\overline{n-3}A^2} - \frac{x}{\overline{n-5}A} = \frac{\overline{n-4}A^2+t^2-\overline{n-3}Ax}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\right)\times\left(A^2+t^2-2Ax\right)^{\frac{3-n}{2}}.$$

By substituting $-t$ and t for x in the preceding expression the attraction of the whole globe on the particle P will be found $\frac{p}{n-1}\left(\frac{A^2+\overline{n-3}At+t^2}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\times\overline{A+t}^{3-n}(L) - \frac{A^2-\overline{n-3}At+t^2}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\times\overline{A-t}^{3-n}(H)\right)$.

16. If the particle be situated within the globe, and consequently A less than t or $-t$, and n be an odd number negative

or

or affirmative; or fraction, of which the numerator is odd; then the preceding resolution will be physically just; for in this case, if the attraction on one side of the particle be affirmative, the attraction on the other will mathematically be negative, that is, physically opposite: but if n be an even number, or fraction, of which the numerator is an even and denominator an odd number, the mathematical solution will not agree to the physical; for in the former the force on both sides will be affirmative, in the latter the forces will be opposite, and therefore physically the force in this case will vary as $L+H$, and not as $L-H$, which is the force in both the cases when A is greater than t and $-t$. The same may be applied to the more general resolution.

17. Let ABCD (fig. 4.) be a globe, of which the diameters AB and CD are situated at right angles to each other, and AHBL be a spheroid generated by the revolution of an ellipse on its axis AB, to which let HL nearly equal to $CD = AB$ be the conjugate, and P a point in the axis BA produced; to find the attraction of the ring contained between the globe and the spheroid on the point P, on the supposition that the force of any corpuscle in the ring on the particle P varies as the magnitude of the corpuscle directly, and the nth power of its distance from the corpuscle inversely.

Let $AB = CD = 2t$, $CD = HL\ (2c) + 2CH = 2c + 2e$, and consequently $OH = c = t - e$, where e has a very small ratio to t, $oP = A$, $po = x$, and pM parallel to $CD = y$; then, by the preceding lemma, the attraction of the circle whose radius is pM on the point P will be $\frac{p \times \overline{A-x}}{\overline{n-1}\,Pp^{n-1}} - \frac{p \times \overline{A-x}}{\overline{n-1}\,PM^{n-1}}$; and in like manner the attraction of the circle, whose radius is pm,

will be $\frac{p\times\overline{A-x}}{\overline{n-1}Pp^{n-1}} - \frac{p\times\overline{A-x}}{\overline{n-1}\,.\,Pm^{n-1}}$, and consequently the attraction of the ring contained between the two circles is $\frac{p\times\overline{A-x}}{\overline{n-1}\,.\,Pm^{n-1}}$

$$-\frac{p\times\overline{A-x}}{\overline{n-1}\,.\,PM^{n-1}} = \frac{p}{n-1}\times\left(\frac{A-x}{\left(\overline{A-x}^2+\frac{\overline{t-e}^2}{t^2}\times\overline{t^2-x^2}\right)^{\frac{n-1}{2}}}-\right.$$

$$\left.\frac{A-x}{\left(\overline{A-x}^2+t^2-x^2\right)^{\frac{n-1}{2}}}\right) = \frac{p}{n-1}\times\frac{\overline{n-1}\times\overline{A-x}\times\left(t-\frac{x^2}{t}\right)}{\left(A^2+t^2-2Ax\right)^{\frac{n+1}{2}}}\times e + \&c.\times e^2 +$$

&c. $= p\times\left(t-\frac{x^2}{t}\right)\times\frac{A-x}{\left(A^2+t^2-2Ax\right)^{\frac{n+1}{2}}}\times e$ nearly, which multiplied into $\dot{x}$, and the fluent of the resulting fluxion found, it will be $p\times\left(A^2+t^2-2Ax\right)^{\frac{1-n}{2}}\times\left(\left(\frac{1}{\overline{n-7}\times tA}=d\right)x^3-\left(\frac{1}{\overline{n-5}t}+\right.\right.$ $\left.\frac{3d\times\overline{A^2+t^2}}{\overline{n-5}A}=c\right)x^2-\left(\frac{t}{\overline{n-3}A}+2c\times\frac{A^2+t^2}{\overline{n-3}\times A}=b\right)x+\frac{t}{n-1}-\frac{b\times\overline{A^2+t^2}}{\overline{n-1}A}\Big)$ e(M).

If the attraction of the whole ring contained between the sphere and spheroid be required, substitute in the fluent (M) for x, t and $-t$, and proceed as in the preceding cases; in like manner may be found the attraction of the ring contained between any two values of x.

18. The same principles may be applied to find the attraction of the above-mentioned ring, when the line Pp is not perpendicular to the circles pM, pm, CD, &c., and does not cut the diameters CD, M′M, &c. into two equal parts. They may be further applied for finding the attraction of rings contained between any other given solids, of which the equations differ by very small quantities from each other; for example,

between two spheroids, of which the axes in the one do not differ much from correspondent axes in the other, in which case the fluents found of the following fluxions may be useful.

$$1.\ \int \frac{\dot{x}}{(t^2-x^2)^{\frac{n-1}{2}}} = \frac{1}{n-3t^2} \times \frac{x}{(t^2-x^2)^{\frac{n-3}{2}}} + \frac{n-4}{\overline{n-3}\,.\,\overline{n-5}\,t^4} \times \frac{x}{(t^2-x^2)^{\frac{n-5}{2}}}$$

$$+ \frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,t^6} \times \frac{x}{(t^2-x^2)^{\frac{n-7}{2}}} + \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,t^8} \times$$

$$\frac{x}{(t^2-x^2)^{\frac{n-9}{2}}} \ldots\ldots \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots 3}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots 2t^{n-3}} \times \frac{x}{t^2-x^2} +$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots 3\,.\,1}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots 2t^{n-3}} \times \int \frac{\dot{x}}{t^2-x^2}$$ if n be an odd number; but

$$2.\ \int \frac{\dot{x}}{(t^2-x^2)^{\frac{n-1}{2}}} = \frac{1}{n-3t^2} \times \frac{x}{(t^2-x^2)^{\frac{n-3}{2}}} + \frac{n-4}{\overline{n-3}\,.\,\overline{n-5}\,t^4} \times \frac{x}{(t^2-x^2)^{\frac{n-5}{2}}}$$

$$+ \frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,t^6} \times \frac{x}{(t^2-x^2)^{\frac{n-7}{2}}} \ldots\ldots +$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots 4\,.\,2}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\ldots 3\,.\,1t^{n-2}} \times \frac{x}{(t^2-x^2)^{\frac{1}{2}}},$$ if n be even.

These principles may be applied to the finding approximations in very many philosophical problems.

Cor. 1. From hence may be deduced the subsequent arithmetical theorems.

$$1.\ \frac{1}{2m-1} + \frac{2m-2}{\overline{2m-1}\,.\,\overline{2m-3}} + \frac{\overline{2m-2}\,.\,\overline{2m-4}}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}} +$$

$$\frac{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}} \ldots\ldots \frac{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}\ldots 2}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\ldots 3\,.\,1}$$

$= 1$; or, which is the same, $\frac{1}{2m-3} + \frac{2m-4}{\overline{2m-3}\,.\,\overline{2m-5}} +$

$2m-4$

$$\frac{\overline{2m-4}\times\overline{2m-6}}{2m-3\times 2m-5\times 2m-7}\ldots\ldots+\frac{\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\ldots 4\times 2}{2m-3\,.\,2m-5\,.\,2m-7\,.\,2m-9\ldots 3\times 1}$$

$$=1,\text{ or }\frac{1}{2m-5}+\frac{2m-6}{2m-5\,.\,2m-7}\ldots+\frac{\overline{2m-6}\,.\,\overline{2m-8}\ldots 4\times 2}{2m-5\,.\,2m-7\,.\,2m-9\ldots 3\times 1}$$

$=1$, &c.

2. $$\frac{1}{2m-2}+\frac{2m-3}{2m-2\,.\,2m-4}+\frac{\overline{2m-3}\,.\,\overline{2m-5}}{2m-2\,.\,2m-4\,.\,2m-6}+$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}}{2m-2\,.\,2m-4\,.\,2m-6\,.\,2m-8}\ldots\ldots+$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots 7\,.\,5}{2m-2\,.\,2m-4\,.\,2m-6\,.\,2m-8\,.\,2m-10\ldots 6\,.\,4}+2\times$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots 3}{2m-2\,.\,2m-4\,.\,2m-6\,.\,2m-8\,.\,2m-10\ldots 2}=1,\text{ or}$$

$$\frac{1}{2m-4}+\frac{2m-5}{2m-4\,.\,2m-6}+\frac{\overline{2m-5}\,.\,\overline{2m-7}}{2m-4\,.\,2m-6\,.\,2m-8}\ldots\ldots+$$

$$\frac{\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots 5}{2m-4\,.\,2m-6\,.\,2m-8\ldots 4}+2\times$$

$$\frac{\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\,.\,\overline{2m-11}\ldots 5\,.\,3}{2m-4\,.\,2m-6\,.\,2m-8\,.\,2m-10\,.\,2m-12\ldots 4\,.\,2}=1,\ \&c.$$

3. By expanding the terms of the preceding fluents may be deduced the arithmetical theorems, *viz.*

1. $$\frac{1}{n-3}\times\overline{n-3}\times\overline{n-1}\,.\,\overline{n+1}\,.\,\overline{n+3}\ldots\ldots\overline{n+m}+\frac{n-4}{n-3\,.\,n-5}$$

$$\times\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\,.\,\overline{n+1}\ldots\overline{n+m-2}+\frac{\overline{n-4}\,.\,\overline{n-6}}{n-3\,.\,n-5\,.\,n-7}$$

$$\times\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\ldots\overline{n+m-4}+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}}{n-3\,.\,n-5\,.\,n-7\,.\,n-9}\times\overline{n-9}\,.\,\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\ldots\overline{n+m-6}$$

$$+\ldots\ldots+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots\ldots 3}{n-3\,.\,n-5\,.\,n-7\,.\,n-9\,.\,n-11\ldots 2}\times 2\,.\,4\,.\,6\,.\,8\ldots$$

$$\overline{m+5}+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots 3}{n-3\,.\,n-5\,.\,n-7\,.\,n-9\,.\,n-11\ldots 2}\times 2\times 4\times 6\times 8\ldots$$

$m+5$

$\overline{m+5}\times\frac{1}{m+6}=\overline{n-1}\cdot\overline{n+1}\cdot\overline{n+3}\cdot\overline{n+5}\dots\overline{n+m+2}\times\frac{1}{m+6}$

if n be an odd number.

2. $\frac{1}{n-3}\times\overline{n-3}\cdot\overline{n-1}\cdot\overline{n+1}\cdot\overline{n+3}\dots\overline{n+2m+1}+$
$\frac{n-4}{\overline{n-3}\cdot\overline{n-5}}\times\overline{n-5}\cdot\overline{n-3}\cdot\overline{n-1}\cdot\overline{n+1}\dots\overline{n+2m-1}+$
$\frac{\overline{n-4}\cdot\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-7}\cdot\overline{n-5}\cdot\overline{n-3}\cdot\overline{n-1}\dots\overline{n+2m-3}+$
$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-9}\cdot\overline{n-7}\cdot\overline{n-5}\cdot\overline{n-3}\dots\overline{n+2m-5}$
$\dots+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\dots2}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\dots3\times1}\times1\cdot3\cdot5\cdot7\dots\overline{2m+5}=$
$\overline{n-1}\cdot\overline{n+1}\cdot\overline{n+3}\cdot\overline{n+5}\dots\overline{n+2m+3}\times\frac{1}{2m+7}$, if n be an even number.

4. Arithmetical theorems, ſomewhat different, may be deduced from taking the fluxions of the preceding fluents, and reducing the fractions reſulting to a common denominator by multiplying them into t^2-x^2, $(t^2-x^2)^2$, $(t^2-x^2)^3$, &c.

1. $\frac{1}{n-3}\times\overline{n-4}+\frac{n-4}{\overline{n-3}\cdot\overline{n-5}}\times\overline{n-7}+\frac{\overline{n-4}\cdot\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-10}$
$+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-13}\dots+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\dots3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\dots2}$
$\times\frac{7-n}{2}-\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\dots3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\cdot\overline{n-11}\dots2}\times\frac{n-3}{2}=0$, if n be odd.

2. $\frac{1}{n-3}\times\overline{n-4}+\frac{n-4}{\overline{n-3}\cdot\overline{n-5}}\times\overline{n-7}+\frac{\overline{n-4}\cdot\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-10}$
$+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-13}\dots+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\dots2}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\dots3\times1}$
$\times\frac{4-n}{2}=0$, if n be even; or more general, if n be even,

3.

$$3.\ \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots\overline{n-2m-2}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots\overline{n-2m-3}}\times\overline{n-2m-4}+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots\overline{n-2m-4}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\ldots\overline{n-2m-5}}\times(\overline{m+1}\times\overline{n-2m-6}-1)$$

$$+\ \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\,.\,\overline{n-12}\ldots\overline{n-2m-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\,.\,\overline{n-13}\ldots\overline{n-2m-7}}\times$$

$$(\overline{m+1}\,.\,\frac{m+2}{2}\times\overline{n-2m-8}-\overline{m+2})+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\,.\,\overline{n-12}\,.\,\overline{n-14}\ldots\overline{n-2m-8}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\,.\,\overline{n-13}\,.\,\overline{n-15}\ldots\overline{n-2m-9}}$$

$$\times\ (\overline{m+1}\,.\,\frac{m+2}{2}\,.\,\frac{m+3}{3}\times\overline{n-2m-10}-\overline{m+2}\,.\,\frac{m+3}{2})\ +$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots\overline{n-2m-10}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\ldots\overline{n-2m-11}}\times(\overline{m+1}\,.\,\frac{m+2}{2}\,.\,\frac{m+3}{3}\,.$$

$$\frac{m+4}{4}\times\overline{n-2m-12}-\overline{m+2}\,.\,\frac{m+3}{2}\,.\,\frac{m+4}{3})\ \ldots\ldots+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots 2}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\ldots 3\times 1}\times-\overline{m+2}\,.\,\frac{m+3}{2}\,.\,\frac{m+4}{3}\ldots\ldots\frac{n-4}{n-2m-6}$$

$$=0.$$

$$4.\ \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots\overline{n-2m-2}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots\overline{n-2m-3}}\times\overline{n-2m-4}+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots\overline{n-2m-4}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots\overline{n-2m-5}}\times(\overline{m+1}\times\overline{n-2m-6}-1)+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots\overline{n-2m-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots\overline{n-2m-7}}\times(\overline{m+1}\,.\,\frac{m+2}{2}\times\overline{n-2m-8}-$$

$$\overline{m+2})+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots\overline{n-2m-8}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots\overline{n-2m-9}}\times(\overline{m+1}\,.\,\frac{m+2}{2}\times$$

$$\frac{m+3}{3}\times\overline{n-2m-10}-\overline{m+2}\,.\,\frac{m+3}{2})+\ldots\ldots\ldots+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots 5\times 3}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots 4\times 2}\times(\overline{m+1}\,.\,\frac{m+2}{2}\,.\,\frac{m+3}{3}\ldots\frac{n-5}{n-2m-5}-$$

$$\overline{m+2}\,.\,\frac{m+3}{2}\,.\,\frac{m+4}{3}\ldots\frac{n-5}{n-2m-7})-$$

$n-4$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots5\times3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots4\times2}\times\overline{m+2}\cdot\frac{m+3}{2}\times\frac{m+4}{3}\ldots\frac{n-3}{n-2m-5}=0,$$ if n be an odd number.

Many more arithmetical theorems may be deduced from the fluents of these and other fluxions by similar methods, which cannot, without some difficulty, be found from the common methods of finding the sums of series.

This method of finding approximations to the areas and lengths of curves, fluents of fluxions, and sums of series, &c. of which the equations, to their increments, fluxions, &c. are given from the areas of curves, fluents, &c. of which the equations to their increments, &c. differ by very small quantities from the given equations was published in the Meditat. Analyt. near twenty years ago.

I shall conclude this Paper with two theorems of some little use in the doctrine of chances.

THEOREM I.

$$H=\overline{a+b}\times\overline{a+b-1}\cdot\overline{a+b-2}\cdot\overline{a+b-3}\ldots\overline{a+b-n+1}$$
$$=a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+1}+n\cdot a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a\quad n+2}$$
$$\times b+n\cdot\frac{n-1}{2}\times a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+3}\times b\cdot\overline{b-1}+n\cdot$$
$$\frac{n-1}{2}\cdot\frac{n-2}{3}a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+4}\times b\times\overline{b-1}\cdot\overline{b-2}+$$
$$\ldots\ldots+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\ldots\frac{n-l+1}{l}\,(\mathrm{L})\,a\cdot\overline{a-1}\cdot\overline{a-2}$$
$$\cdot\overline{a-3}\ldots\overline{a-n+l+1}\times b\cdot\overline{b-1}\cdot\overline{b-2}\ldots\overline{b-l+1}+\ldots+$$

 n.

$n \cdot \frac{n-1}{2} a \cdot \overline{a-1} \cdot b \cdot \overline{b-1} \cdot \overline{b-2} \cdot\cdot \overline{b-n+3} + na \cdot b \cdot \overline{b-1} \cdot \overline{b-2} \cdots \overline{b-n+2} + b \cdot \overline{b-1} \cdot \overline{b-2} \cdots \overline{b-n+1}$.

If for $a+b-1$, $a+b-2$, $a+b-3$, &c., $a-1$, $a-2$, &c., $b-1$, $b-2$, &c. be ſubſtituted reſpectively $a+b-x$, $a+b-2x$, $a+b-3x$, &c., $a-x$, $a-2x$, $a-3x$, &c., $b-x$, $b-2x$, $b-3x$, &c., the reſulting equation will equally be juſt; and, laſtly, if for x be ſubſtituted o, it will become the binomial theorem.

Cor. If there are two different events A and B, of which the numbers are reſpectively a and b, and their chances of happening alſo as a and b; and if A's happen, let the whole number $(a+b)$ and alſo the number of A's be diminiſhed by x, and in the ſame manner of B's happening, and ſo on; then will the chance of A's happening $n-l$ times, and B's happening l times in n trials be $L \times a \cdot \overline{a-x} \cdot \overline{a-2x} \cdot\cdot \overline{a-(n-l-1)x} \times b \cdot \overline{b-x} \cdot \overline{b-2x} \cdot\cdot \overline{b-(l-1)x}$ divided by H.

In a ſimilar manner may be found, 1. the chance of A's happening between h and k times; and, 2. the chance of A's happening (h) to B's happening (k) times; 3. of A's and B's happening reſpectively h and k times more than the other; 4. the chance of A's happening an even to its happening an odd number of times, &c. in (n) trials, &c. &c. &c.

THEOREM II.

$$H = \overline{a+b+c+d+\&c.} \times \overline{a+b+c+d+\&c.-x} \times \overline{a+b+c+d+\&c.-2x} \cdots \overline{a+b+c+d+\&c.-(n-1)x} = a \cdot \overline{a-x} \cdot \overline{a-2x} \cdots \overline{a-\overline{n-1}x} + n \cdot a \cdot \overline{a-x} \cdot \overline{a-2x} \cdot\cdot \overline{a-\overline{n-2}x} \times \overline{b+c+d+\&c.}$$

+

$+ n \cdot \frac{n-1}{2} \cdot a \cdot \overline{a-x} \cdot \overline{a-2x} \ldots \overline{a-n-3x} \times (b \cdot \overline{b-x} + c \times \overline{c-x}$
$+ d \cdot \overline{d-x} + \&c. + 2bc + 2bd + 2cd + \&c.) + \ldots + \mathrm{L} \times (a \cdot$
$\overline{a-x} \cdot \overline{a-2x} \ldots \overline{a-l-1x} \times b \cdot \overline{b-x} \cdot \overline{b-2x} \ldots \overline{b-m-1x}$
$\times c \cdot \overline{c-x} \cdot \overline{c-2x} \ldots \overline{c-p-1x} \times d \cdot \overline{d-x} \cdot \overline{d-2x} \ldots$
$\overline{d-q-1x} \times \&c. = \mathrm{K}) + \&c.$, where $\mathrm{L} = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \ldots$
$\frac{n-l+1}{l} \times \overline{n-l} \cdot \frac{n-l-1}{2} \cdot \frac{n-l-2}{3} \ldots \frac{n-l-m+1}{m} \times \overline{n-l-m} \cdot$
$\frac{n-l-m-1}{2} \cdot \frac{n-l-m-2}{3} \ldots \frac{n-l-m-p+1}{p} \times \overline{n-l-m-p} \cdot$
$\frac{n-l-m-p-1}{2} \cdot \frac{n-l-m-p-2}{3} \ldots \frac{n-l-m-p-q+1}{q} \times \&c.$ which is the ſame as the co-efficient of the term $a^l \times b^m \times c^p \times d^q \times \&c.$ in the multinomial $a + b + c + d + \&c.$ raiſed to the power n.

The chance of any number of events A, B, C, D, &c. of which the numbers are a, b, c, d, &c. happening l, m, p, q, &c. times reſpectively in a ſimilar manner to A's and B's happening in the preceding caſe will be $\frac{\mathrm{L} \times \mathrm{K}}{\mathrm{H}}$.

All the propoſitions mentioned as immediately deducible from the preceding theorem may, *mutatis mutandis*, with the ſame eaſe be applied to more events A, B, C, D, &c.

If for a, b, c, d, &c. be ſubſtituted the ſame letters, increaſed or diminiſhed by any given quantities, the reſulting equation will be equally true.

Fig 1.

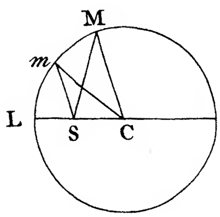
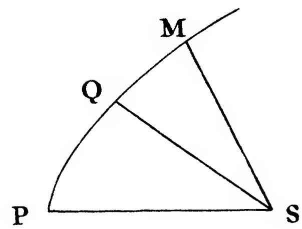


Fig 2.



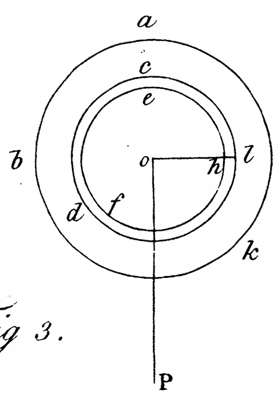


Fig 3.

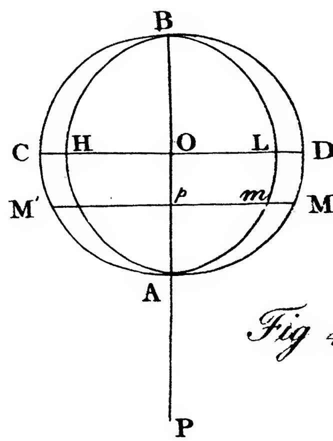


Fig 4.